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SOME FIXED POINT THEOREMS ON EXTENDED C´ IRIC´ CONTRACTION IN QUASI-PARTIAL B-METRIC SPACE

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Abstract

The aim of this study is to obtain fixed points by adopting the approach of extended C´ iric´ contraction mapping in the notion of complete quasi-partial b-metric space. Furthermore, we have extended the Bota's Theorem and es- tablished the corresponding fixed point results in the setting of quasi-partial-b metric space. Our result is supported with a suitable example.

Keywords: fixed point; C´ iric´ - contraction; Bota's - contraction; quasi partial b metric

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1 Introduction and Preliminaries

Metric fixed point theory was first developed by the renowned mathematician Ba- nach [1] who commenced the pivotal result named Banach Contraction Principle. This result has an extensive application in finding the unique solution of certain integral equation. i.e., Consider a self mapping S on a non-empty set U. Let d be a complete metric on U. If there exists a constant $\rho \in [0, 1)$ s. t.

$$d(S\xi, Sv) \leq \rho d(\xi, v) \qquad \text{for all } \xi, v \in U,$$

then S possesses a unique fixed point in X. Many researchers defined the various other forms of new contractive conditions and generalized new spaces in different fields. See [2, 3, 4]. One of prominent space is partial metric space which was presented by Matthews [5] in 1994. Later on, several authors obtained generalized version of celebrated Banach contraction principle. See [6, 7, 8]. As we know the fact that in Banach contraction principle, self map S is continuous which is con- sidered to be a weakness of the theorem. To remove this superfluous condition of

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continuity, Kannan [9] introduced a new mapping known as a Kannan contraction i.e.,

$d(S\xi, Sv) \leq \rho[d(\xi, S\xi) + d(v, Sv)]$ for all $\xi, v \in U$,

where $\rho \in [0, \frac{1}{2})$

In 1968, Bryant [10] introduced a new concept in contraction i.e., A map S need not have to be a contraction, but for some $n \in N$, the map S^n may be a contraction. Sehgal [11] extended the notion and established a unique fixed point. i.e., Consider a complete metric space (U,d) and $\alpha \in [0,1)$ and a self map $S : U \to U$ be a continuous map. If for each $\xi \in U$ there exists a positive integer $k = k(\xi)$ such that

$$d(S^{k(\xi)}\xi, S^{k(\xi)}v) \leq \alpha d(\xi, v) \quad \text{for all } \xi, v \in U.$$

Hence S is a unique fixed point in U. On expansion of contractive maps, in 1972, Reich [12] introduced a new class of mappings which is a generalisation of the Kannan contraction and Banach contraction, e.g., a self mapping $S : U \to U$ is called a Reich-contraction if there are $\alpha_1, \alpha_2, \alpha_3 \in [0, 1)$ and $\alpha_1 + \alpha_2 + \alpha_3 < 1$ such that

$$d(S\xi, Sv) \leq \alpha_1 d(\xi, S\xi) + \alpha_2 d(v, Sv) + \alpha_3 d(\xi, v) \quad \text{for all } \xi, v \in U.$$

A self map $S : U \to U$ is called a Reich–Rus–C´ iric´ contraction mapping on a complete metric space (U, d) if there are $\rho \in [0, \frac{1}{3})$ such that

$$d(S\xi, Sv) \leq \rho[d(\xi, v) + d(\xi, S\xi) + d(v, Sv)],$$

for all $\xi, v \in X$, then S possesses a unique fixed point. See [13, 14, 15, 16, 17, 18, 19, 20]. As a generalisation of spaces, Gupta and Gautam [20, 21] defined quasi- partial b -metric space(QPBMS) and established fixed point results on this space. Since then, many authors have contributed in the development of metric fixed point theory. [22, 23, 24, 25, 26, 27, 28]. For further study related to this field see([29, 30, 31, 32, 33, 34].

In this paper, we have proved the existence of fixed point in extended C´ iric´ con- traction and Bota's contraction in this space.

Let us recall the basic definitions of a QPBMS.

Definition 1.1 ([20]) Let (U, qp_b) is a QPBMS where U is a non-empty set and qp_b defined as $qp_b : U \times U \to R^+$ such that for some real number $s \geq 1$ and all $\xi, v, z \in U$:

1. $qp_b(\xi, \xi) = qp_b(\xi, v) = qp_b(v, v)$ implies $\xi = v$,
2. $qp_b(\xi, \xi) \leq qp_b(\xi, v)$,
3. $qp_b(\xi, \xi) \leq qp_b(v, \xi)$,
4. $qp_b(\xi, v) \leq s[qp_b(\xi, z) + qp_b(z, v)] - qp_b(z, z)$. Here s is defined as a coefficient of (U, qp_b).

Let qp_b be a QPBM on the set U . Then

$$d_{qp_b}(\xi, v) = qp_b(\xi, v) + qp_b(v, \xi) - qp_b(\xi, \xi) - qp_b(v, v) \text{ is a b-metric on } X.$$

Lemma 1.1 ([21]) Let (U, qp_b) be a QPBMS. Then:

1. If $qp_b(\xi, \nu) = 0$ then $\xi = \nu$.

2. If $\xi \neq \nu$, then $qp_b(\xi, \nu) > 0$ and $qp_b(\nu, \xi) > 0$.

Definition 1.2 ([21]) Let us consider a QPBM (U, qp_b). Then

1. a sequence $\{\xi_n\} \subset U$ converges to $\xi \in U$ iff

$$qp_b(\xi, \xi) = \lim_{n\to\infty} qp_b(\xi, \xi_n) = \lim_{n\to\infty} qp_b(\xi_n, \xi).$$

2. a sequence $\{\xi_n\} \subset U$ is said to be a Cauchy sequence iff

$\lim_{n,m\to\infty} qp_b(\xi_n, \xi_m)$ and $\lim_{m,n\to\infty} qp_b(\xi_m, \xi_n)$ exists (and are finite).

3. The QPBMS (U, qp_b) is said to be complete if every Cauchy sequence $\{\xi_n\} \subset U$ converges with respect to τ_{qpb} to a point $\xi \in X$ such that

$$qp_b(\xi, \xi) = \lim_{n,m\to\infty} qp_b(\xi_n, \xi_m) = \lim_{m,n\to\infty} qp_b(\xi_m, \xi_n).$$

4. A map $g : U \to U$ is continuous at $\xi_0 \in U$ if, for every $\varepsilon > 0$, there exist

$$\delta > 0 \text{ such that } g(B(\xi_0, \delta)) \subset B(g(\xi_0), \varepsilon).$$

Lemma 1.2 ([23]) Consider (U, qp_b) be a QPBMS and (U, d_{qpb}) be the corre- sponding b-metric space. Then (U, d_{qpb}) is complete if (U, qp_b) is complete.

Lemma 1.3 ([24]) Let (U, qp_b) be a QPBMS and $S : U \to U$ be a given map. S is called a sequentially continuous at $z \in U$ if for each sequence $\{\xi_n\}$ in U converging to z, we have: $S\xi_n \to Sz$, i.e., $qp_b(S\xi_n, Sz) = qp_b(Sz, Sz)$.

2 Main Results

We start this section by the following result.

Theorem 2.1 Let us consider (U, qp_b) be a complete QPBMS with $s \geq 1$ and $S : U \to U$ be a self map. If for each $\xi \in U$ there exists a positive integer $n = n(\xi)$ such that

$$qp_b(S^n\xi, S^n\nu) \leq \alpha \max\{qp_b(\xi, \nu), qp_b(\xi, S\nu), qp_b(\xi, S^2\nu), \ldots, d(\xi, S^n\nu), d(\xi, S^n\xi)\} \quad (2.1)$$

satisfies for some $\alpha \in [0, \frac{1}{s})$ and all $\nu \in U$, then S has a unique fixed point $\xi \in U$. Moreover, for every $\xi \in U$ we get $\lim_{m\to\infty} S^m\xi = \xi^*$.

First, we will show the orbit, $\{S^m\xi\}_{m=0}^{\infty}$, is bounded for all $\xi \in U$. Let us prove that,

$$r(\xi) = \sup\{qp_b(\xi, S^m\xi): m \in N\} \le \frac{1}{1 - s\alpha} \max\{qp_b(\xi, S^q\xi): 0 < q \le n(\xi)\}, \quad (2.2)$$

for any $\xi \in U$. Let m is any positive integer and k is a positive integer which depends on $\xi \in U$ and m such that

$$qp_b(\xi, S^k\xi) = \max\{qp_b(\xi, S^p\xi): 0 < p < m\}. \quad (2.3)$$

Let us assume that $k, m > n$. Hence from 2.1 we get

$$\begin{aligned} qp_b(\xi, S^k\xi) &\le s[qp_b(\xi, S^n\xi) + qp_b(S^n\xi, S^nS^{k-n}\xi)] \\ &\le s[qp_b(\xi, S^n\xi) + \alpha \max\{qp_b(\xi, S^{k-n}\xi), qp_b(\xi, S^{k-n+1}\xi), \\ &\quad \dots, qp_b(\xi, S^k\xi), qp_b(\xi, S^n\xi)\}] \\ &\le s\, qp_b(\xi, S^n\xi) + s\,\alpha \max\{qp_b(\xi, S^p\xi): 0 < p < m\} \end{aligned}$$

By using 2.3, we get $qp_b(\xi, S^k\xi) \le s\, qp_b(\xi, S^n\xi) + s\,\alpha\, qp_b(\xi, S^k\xi)$ and there- fore

$$qp_b(\xi, S^k\xi) \le s\, qp_b(\xi, S^n\xi)/1 - s\alpha.$$

Since m is arbitrary we will say

$$\sup_{m>n(\xi)} qp_b(\xi, S^m\xi) \le qp_b(\xi, S^k\xi) \le s\, qp_b(\xi, S^{n(\xi)}\xi)/1 - s\alpha,$$

and therefore 2.2 satisfies. Next, $\{S^m\xi\}_{m=0}^{\infty}$, is bounded for every $\xi \in U$. Now next, we shall prove that the sequence $\{S^m\xi_0\}$ is Cauchy, where $\xi_0 \in U$ is an arbitrary. For this aim, we set up a sub-sequence $\{\xi_k\}$: choosing arbitrary point $\xi_0 \in U$ with $n_0 = n(\xi_0)$, we set $\xi_1 = S^{n_0}\xi_0$ and by induction we get

$$\xi_{i+1} = S^{n_i}\xi_i \text{ with } n_i = n(\xi_i).$$

We choose any arbitrary $\xi_k \in \{\xi_k\}$ and we fixed it. Let $\xi_p = S^p\xi_0$, $\xi_q = S^q\xi_0$ be two members of $\{S^m\xi_0\}$ that are successor terms of ξ_k. Then $\xi_p = S^u\xi_k$ and $\xi_q = S^v\xi_k$ for some u, v respectively. Then by 2.1 we conclude

$$\begin{aligned} qp_b(\xi_k, \xi_p) &= qp_b(S^{n_{k-1}}\xi_{k-1}, S^u\xi_k) \\ &= qp_b(S^{n_{k-1}}\xi_{k-1}, S^{n_{k-1}}S^{u-n_{k-1}}\xi_{k-1}) \\ &\le \alpha \max\{qp_b(\xi_{k-1}, S^{u-n_{k-1}}\xi_{k-1}), qp_b(\xi_{k-1}, S^{u-n_{k-1}+1}\xi_{k-1}), \\ &\quad \dots, qp_b(\xi_{k-1}, S^u\xi_{k-1}), qp_b(\xi_{k-1}, S^{n_{k-1}}\xi_{k-1})\} \\ &= \alpha\, qp_b(\xi_{k-1}, S^{u_1}\xi_{k-1}), \end{aligned}$$

where $u_1 \in \{u - n_{k-1}, u - n_{k-1} + 1, \dots, u, n_{k-1}\}$ such that

$$\begin{aligned} qp_b(\xi_{k-1}, S^{u_1}\xi_{k-1}) = \max\{&qp_b(\xi_{k-1}, S^{u-n_{k-1}}\xi_{k-1}), qp_b(\xi_{k-1}, S^{u-n_{k-1}+1}\xi_{k-1}) \\ &, \dots, qp_b(\xi_{k-1}, S^u\xi_{k-1}), qp_b(\xi_{k-1}, S^{n_{k-1}}\xi_{k-1})\} \end{aligned} \quad (2.4)$$

Continuing this process, we get

$qp_b(\xi_{k-1}, S^{u1}\xi_{k-1}) \leq \alpha \max\{qp_b(\xi_{k-2}, S^{u1}\xi_{k-2}, \ldots, qp_b(\xi_{k-2}, S^{nk-2}\xi_{k-2})\}$
$= \alpha\, qp_b(\xi_{k-2}, S^{u2}\xi_{k-2})$.

On computing k-times, we have

$qp_b(\xi_k, \xi_p) \leq \alpha qp_b(\xi_{k-1}, S^{u1}\xi k - 1) \leq \alpha^2 qp_b(\xi_{k-2}, S^{u2}\xi_{k-2}) \leq \ldots$
$\leq \alpha^k qp_b(\xi_0, S^{uk}\xi_0)$.

Consequently, we obtain that

$$qp_b(\xi_k, \xi_p) \leq \alpha^k\, r(\xi_0).$$

Analogously, we also get that

$$qp_b(\xi_k, \xi_q) \leq \alpha^k\, r(\xi_0).$$

By using the definition of QPBMS, we derive that

$$qp_b(\xi_p, \xi_q) \leq s[qp_b(\xi_k, \xi_p) + qp_b(\xi_k, \xi_q)] \leq 2s\alpha^k r(\xi_0). \quad (2.5)$$

So, we prove that the orbit $\{S^m\xi_0\}$ is a Cauchy. As (U, qp_b) is a complete QPBMS and there is a ξ^* $\in X$ such that $\xi^* = \lim_{m\to\infty} S^m\xi_0$. Now next, we will prove that ξ^* is a fixed point of $S^n(\xi^*)$. Let $m \geq n = n(\xi^*)$,

$$\begin{aligned} qp_b(\xi^*, S^n\xi^*) &\leq s[qp_b(\xi^*, S^m\xi_0) + qp_b(S^n\xi^*, S^nT^{m-n}\xi_0)] \\ &\leq s[qp_b(\xi^*, S^m\xi_0) + \alpha \max\{qp_b(\xi^*, S^{m-n}\xi_0), qp_b(\xi^*, S^{m-n+1}\xi_0), \\ &\quad \ldots, qp_b(\xi^*, S^m\xi_0), qp_b(\xi^*, S^n\xi^*)\}]. \end{aligned}$$

On taking the limit as $m \to \infty$,

$$qp_b(\xi^*, S^n\xi^*) \leq \alpha\, qp_b(\xi^*, S^n\xi^*)$$

Since $\alpha \in (0, \frac{1}{s})$, we find that ξ^* is a fixed point of $S^n(\xi^*)$. To prove the unique fixed point, consider ξ^* and v^* be the two distinct fixed points and $n = n(\xi^*)$. We get

$$\begin{aligned} qp_b(\xi^*, v^*) &= qp_b(S^n\xi^*, S^nv^*) \\ &\leq \alpha\max\{qp_b(\xi^*, v^*), qp_b(\xi^*, Sv^*), qp_b(\xi^*, S^2v^*), \\ &\quad \ldots, qp_b(\xi^*, S^nv^*), qp_b(\xi^*, S^n\xi^*)\} \\ &\leq \alpha\, qp_b(\xi^*, v^*) \end{aligned}$$

which gives contradiction, as $\alpha \in (0, \frac{1}{s})$. Uniqueness and $S^{n(\xi^*)}\xi^* = \xi^*$ shows

that ξ^* is another fixed point of S. Say,

$$S\xi^* = SS^{n(\xi^*)}\xi^* = S^{n(\xi^*)}S\xi^*.$$

Hence the proof of this theorem is complete.

Example 2.1 Consider $U = [0, 4]$ defined with QPBMS $qp_b(\xi, v) = |\xi - v| + |\xi|$. Let S be self mapping on QPBM defined by

$$S\xi = \begin{cases} \xi, & \xi \in [0, 2] \\ 1, & \xi \in (2, 4] \end{cases}$$

Then (the point) 0 is a unique fixed point of the map S satisfying equation 2.1 where $n = 2$ and $\alpha \geq 1$.

Case I For $\xi, v \in [0, 2]$ we have,

$$qp_b(S^2\xi, S^2v) = |\xi - v| + |\xi|$$
$$\max\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), qp_b(\xi, S^2\xi)\} = |\xi - v| + |\xi|$$

It implies,

$$qp_b(S^2\xi, S^2v) \leq \alpha \max\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), qp_b(\xi, S^2\xi)\}.$$

Therefore the inequality which is required in equation 2.1 holds for $\xi, v \in [0, 2]$ as shown in Figure 1.

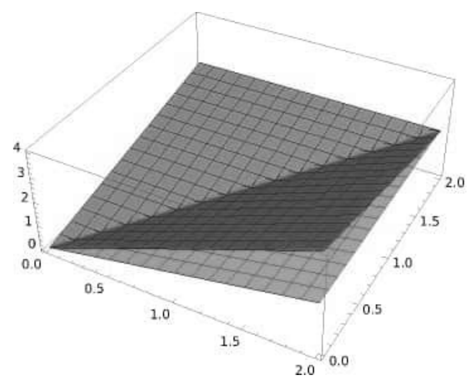

Figure 1: Dominance of right hand side of Equation (2.1) is visually checked for $\xi, v \in [0, 2]$.

Case II For $\xi \in [0, 2]$, $v \in (2, 4]$ we have,

$$qp_b(S^2\xi, S^2v) = |\xi - 1| + |\xi|$$
$$\max\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), qp_b(\xi, S^2\xi)\} = |\xi - 1| + |\xi|$$

It implies,

$$qp_b(S^2\xi, S^2v) \leq \alpha \max\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), qp_b(\xi, S^2\xi)\}.$$

Hence the inequality necessary in equation 2.1 holds for $\xi \in [0, 2]$, $v \in (2, 4]$ as shown in Figure 2.

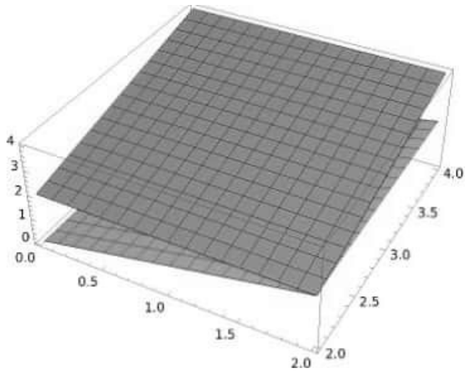

Figure 2: Dominance of right hand side of Equation (2.1) for $\xi \in [0, 2]$, $\nu \in (2, 4]$.

Case III For $\xi \in (2, 4]$, $\nu \in [0, 2]$ we have,

$$qp_b(S^2\xi, S^2\nu) = |1 - \nu| + 1$$
$$\max\{qp_b(\xi, \nu), qp_b(\xi, S\nu), qp_b(\xi, S^2\nu), qp_b(\xi, S^2\xi)\} = |\xi - \nu| + |\xi|$$

It implies,

$$qp_b(S^2\xi, S^2\nu) \leq \alpha \max\{qp_b(\xi, \nu), qp_b(\xi, S\nu), qp_b(\xi, S^2\nu), qp_b(\xi, S^2\xi)\}.$$

Therefore, the inequality mandatory in equation 2.1 holds for $\xi \in (2, 4]$, $\nu \in [0, 2]$ as shown in Figure 3.

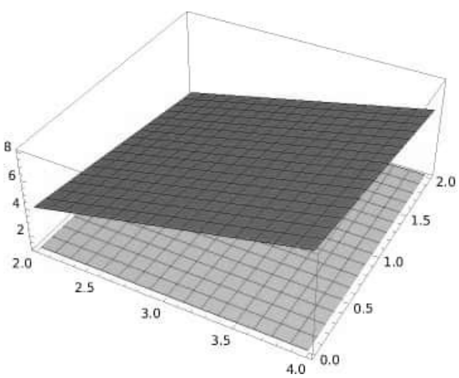

Figure 3: Dominance of right hand side of Equation (2.1) that is visually checked for $\xi \in (2, 4]$, $\nu \in [0, 2]$.

Case IV For $\xi, \nu \in (2, 4]$ we have,

$$qp_b(S^2\xi, S^2\nu) = 1$$
$$\max\{qp_b(\xi, \nu), qp_b(\xi, S\nu), qp_b(\xi, S^2\nu), qp_b(\xi, S^2\xi)\} = |\xi - 1| + |\xi|$$

It implies,

$$qp_b(S^2\xi, S^2v) \leq \alpha\, ma\xi\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), qp_b(\xi, S^2\xi)\}.$$

Therefore, the inequality in equation 2.1 holds for $\xi, v \in (2, 4]$ as shown in Figure 4.

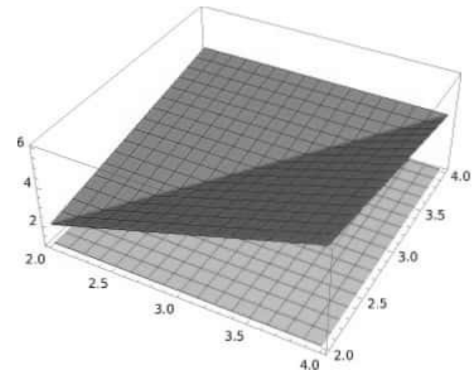

Figure 4: Dominance of right hand side of Equation (2.1) that is visually checked for $\xi, v \in (2, 4]$.

Hence, Theorem 2.1 is satisfied ($n = 2$, $\alpha = 1$) and S has common fixed point 0.

Theorem 2.2 Consider a complete QPBMS (U, qp_b) with $s \geq 1$ and $S : U \to U$ is a map that is continuous. If for each $\xi \in U$ there exists a positive integer $n = n(\xi)$ such that

$$qp_b(S^n\xi, S^nv) \leq \alpha \max\{qp_b(\xi, v), qp_b(\xi, Sv), qp_b(\xi, S^2v), ..., qp_b(\xi, S^nv), qp_b(\xi, S\xi), qp_b(\xi, S^2\xi), ..., qp_b(\xi, S^n\xi)\}, \tag{2.6}$$

holds for some $\alpha \in [0, \frac{1}{s})$ and all $v \in U$, then S has a unique fixed point $\xi^* \in U$. Moreover, for every $\xi \in U$ $\lim_{m\to\infty} S^m\xi = \xi^*$.

By using Theorem 2.1, we conclude that the orbit $\{S^m\xi_0\}$ is bounded and it is Cauchy sequence. Since QPBMS is complete space, it has limit $\xi^* \in U$. Continuity property of S gives us that

$$S^{n(\xi^*)}\xi^* = S^{n(\xi^*)} \lim_{m\to\infty} S^m\xi_0 = \lim_{m\to\infty} S^{m+n(\xi^*)}\xi_0 = \xi^*.$$

Thus, ξ^* is the fixed point of $S^{n(\xi^*)}\xi^*$. Similar to the Theorem 2.1 we get that ξ^* is the unique fixed point of S.

3 Bota's Theorem in QPBMS

In 2016, Bota [22] introduced operators in relation with a contractive iteration in the notion of b metric space. In our next result, we have generalised Bota theorem in notion of QPBMS.

Definition 3.1 Consider a function $\phi : [0, \infty) \to [0, \infty)$ that satisfies the following properties :

(cf_1) ϕ is increasing;

(cf_2) $\lim_{n\to\infty} \phi^n(t) = 0$, for $t \in [0, \infty)$.

Here, Φ be the class of the comparison function $\phi : [0, \infty) \to [0, \infty)$. If ϕ is a comparison function so :

(cf_i) each ϕ^k is a comparison function, for all $k \in \mathbb{N}$;

(cf_{ii}) ϕ is continuous map at 0;

(cf_{iii}) $\phi(t) < t$ for all $t > 0$.

Definition 3.2 A function $\phi_c : [0, \infty) \to [0, \infty)$ is said to be a c-comparison func- tion if :

(ccf_i) ϕ_c is monotone increasing;

(ccf_{ii}) $\sum_{n=0}^{\infty} \phi_c^n(t) < \infty$, for all $t \in (0, \infty)$.

The family of c-comparison functions is denoted by Φ_c.

Definition 3.3 A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a b-comparison func- tion if :

(bcf_1) ϕ is monotone increasing;

(bcf_2) $\sum_{n=0}^{\infty} s^n \phi^n(u) < \infty$, for all $u \in (0, \infty)$ and $s \geq 1$ a real number. We are denoting by Φ_b the family of b-comparison functions.

Notice that any b-comparison function is a comparison function.

Theorem 3.1 Let (X, qp_b, s) be a complete QPBMS with $s \geq 1$ and $S : U \to U$ a map that satisfies the condition : there exists $\phi \in \Phi_b$ such that for each $\xi \in U$ there is a positive integer $n(\xi)$ such that for all $v \in U$

$$qp_b(S^{n(\xi)}(\xi), S^{n(\xi)}(v)) \leq \phi(qp_b(\xi, v)). \tag{3.1}$$

Then, S has a unique fixed point $\xi^* \in U$ and $S^n(\xi_0) \to \xi^*$ for each $\xi_0 \in U$, as $n \to \infty$.

From the initial proof of Theorem 2.1, we conclude that the orbit $\{S^m\xi_0\}$ is bounded. By Theorem 2.1, we complete the proof.

We shall show that the sequence $\{S^m\xi_0\}$ is Cauchy, where $\xi_0 \in U$ be an arbi- trary. Now, we shall construct a sub-sequence $\{\xi_k\}$ in the following way: For an arbitrary point $\xi_0 \in X$ with $n_0 = n(\xi_0)$, we set $\xi_1 = S^{n_0}\xi_0$ and recursively we find

$$\xi_{i+1} = S^{n_i}\xi_i \quad \text{with} \quad n_i = n(\xi_i).$$

We consider any arbitrary ξ_k ξ_k and fixed it. Now take two members $\xi_p = S^p\xi_0$, $\xi^q = S^q\xi_0$ of $S^m\xi_0$ that are successor terms of ξ_k. Then $\xi_p = S^u\xi_k$ and $S^v\xi_k$ for some u, v respectively. Then by 2.1 we conclude

$qp_b(\xi_k, \xi_p) = qp_b(S^{n_{k-1}}\xi_{k-1}, S^u\xi_k)$

$= qp_b(S^{n_{k-1}}\xi_{k-1}, S^{n_{k-1}}S^{u-n_{k-1}}\xi_{k-1})$

$$\leq \phi(qp_b(\xi_{k-1}, S^u\xi_{k-1})$$

$$< qp_b(\xi_{k-1}, S^{u1}\xi_{k-1}).$$

Continuing in this way, we have

$$qp_b(\xi_{k-1}, S^{u1}\xi_{k-1}) \leq \phi(qp_b(\xi_{k-2}, S^{u1}\xi_{k-2}))$$
$$< qp_b(\xi_{k-2}, S^{u2}\xi_{k-2}).$$

Completing this computation k-times we have

$qp_b(\xi_k, \xi_p) \leq \phi(qp_b(\xi_{k-1}, S^{u1}\xi_{k-1})) \leq \phi^2(qp_b(\xi_{k-2}, S^{u2}\xi_{k-2})) \leq \ldots$
$\leq \phi^k(qp_b(\xi_0, S^{uk}\xi_0))$

Consequently, we obtain that

$$qp_b(\xi_k, \xi_p) \leq \phi^k(r(\xi)) < r(\xi).$$

Analogously, we also get that

$$qp_b(\xi_k, \xi_q) \leq \phi^k(r(\xi)) < r(\xi).$$

By using the triangle inequality, we get

$$qp_b(\xi_p, \xi_q) \leq s[qp_b(\xi_k, \xi_p) + qp_b(\xi_k, \xi_q)] \leq 2r(\xi). \quad (3.2)$$

The orbit $\{S^m\xi_0\}$ is a Cauchy.

As (X, qp_b) is a complete QPBMS and there is a $\xi^* \in U$ such that $\xi^* = \lim_{m\to\infty} S^m\xi_0$.

We show that ξ^* is a fixed point of $S^{n(\xi^*)}$. Let $m \geq n = n(\xi^*)$, we have

$$qp_b(S^n\xi^*, S^{n+m}\xi_0) \leq \phi^n(qp_b(\xi^*, S^{m-n}\xi_0))$$

Taking the limit as $m \to \infty$

$$qp_b(\xi^*, S^n\xi^* \leq 0$$

which gives that ξ^* is a fixed point of $S^{n(\xi^*)}$. To show the unique fixed point, consider ξ^* and ν^* are two distinct fixed point and $n = (\xi^*)$. We get

$$qp_b(\xi^*, \nu^*) = qp_b(S^n\xi^*, S^n\nu^*)$$

$\leq \phi(qp_b(\xi^*, \nu^*))$
$< qp_b(\xi^*, \nu^*)$

which contradicts.
Uniqueness and $S^{n(\xi^*)}\xi^* = \xi^*$ gives that ξ^* is also the fixed point of S. Say,

$$S\xi^* = TT^{n(\xi^*)}\xi^* = S^{n(\xi^*)}S\xi^*.$$

4 Conclusions

The major contribution of this manuscript is to prove the existence of unique fixed points in extended C´iric´ contraction map in the setting of quasi-partial b-metric space. Common and coupled fixed points for such type of mappings and their implementation in the field of science and technology will be an impressive concept for future study.

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